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Capacity of M-ary 2-D RLL Constraints for Optical Recording Channels

Hua Hu, Duanyi Xu, Guosheng Qi
Dept. of Precision Instruments & Mechanics
Optical Memory National Engineering Research Center
Tsinghua University, Beijing, 100084, P. R. China
Email: hh01@mails.tsinghua.edu.cn

***Abstract*— A new class of M-ary two-dimensional (2-D) run-length limited (RLL) constraints is introduced in this paper. In particular, only RLL d constraints on 2-D $m \times n$ arrays with limited height m are considered. Column vectors of 2-D arrays are used as states in a directed graph. The transfer matrix of the graph is obtained with consideration of non-binary alphabet, and the largest eigenvalue of the transfer matrix is used to calculate the capacity of 2-D (M, d, ∞) RLL constraints on 2-D arrays. Some numerical results are derived for RLL $d = 1$ and $d = 2$ constraints. We also present an example of efficient finite-state encoder for $d = 1$ constraint by state splitting and merging algorithm. The 2-D (M, d, ∞) constrained codes have potential to be applied in advanced 2-D optical recording channels.**

I. Introduction

Recent developments in page-oriented storage technologies such as holographic memory have motivated the study of two-dimensional (2-D) run-length limited (RLL) codes. The binary 2-D (d, k) RLL constraints require that any two ones are separated by at least d zeros and at most k zeros both horizontally and vertically in an $m \times n$ rectangle. The capacity of a 2-D (d, k) RLL constraint is defined as

$$C = \lim_{m,n\to\infty} \frac{\log_2 N_{m,n}}{mn} \tag{1}$$

where $N_{m,n}$ denotes the total number of $m \times n$ rectangles satisfying the 2-D RLL constraints.

The capacities of different 2-D RLL constraints are not known exactly, but have been very accurately upper- and lower-bounded. An important and special constraint is 2-D $(1, \infty)$ constraint (i.e. hard-square model), which has been studied in terms of capacity estimation [1]-[2] and efficient coding schemes such as 2-D bit-stuffing algorithm [3]-[5]. As an extension of 2-D RLL constraints, checkerboard constraints have also been investigated on the capacity bounds [6]-[7]. In addition to these 2-D constraints, some other constraints are also used to model 2-D recording channels for different applications [8]-[9].

If 2-D optical recording channels permit M-level input signal ($M \geq 3$), higher information density and transfer rate can be achieved. Recently, multi-wavelength multi-level (MWML) optical storage with photochromic media [10]-[11] presents a recording channel requiring M-ary and 2-D RLL constraints. Two-dimensional optical storage (TwoDOS) with multi-level modulation has also been investigated for future high-density optical storage [12]. Although there are increasing research work in binary 2-D RLL constraints, there is very little work related to M-ary 2-D RLL constraints.

In this paper, we introduce a new class of M-ary 2-D RLL constraints for advanced optical recording channels. In particular, we only consider the RLL d constraint. Assuming an M-ary symbol alphabet, $A = \{0, 1, 2, \cdots M - 1\}$, $2 < M < \infty$, the M-ary 2-D RLL (d, ∞) or 2-D (M, d, ∞) constraint is the one where at least d zeros occur between any two nonzero symbols both horizontally and vertically. The proposed constraints may be looked as an expansion of the traditional binary 2-D RLL constraints. Practical 2-D optical recording channels are generally modelled to have finite sub-channels, which limits the corresponding height m of 2-D $m \times n$ arrays to be finite value. Thus we only need to discuss the 2-D (M, d, ∞) constraint on $m \times n$ arrays with fixed height instead of entire plane.

The paper is organized as follows. In section II, we introduce the graph description of 2-D (M, d, ∞) constraints on $m \times n$ arrays, and discuss the capacity of 2-D (M, d, ∞) constraints via transfer matrix method. Some numerical results of capacity are derived in the cases of $d = 1$ and $d = 2$. In section III, we consider the low complexity and efficient encoder/decoder by state splitting and merging algorithm. An example of code construction for $M = 5$ and $d = 1$ is presented. In Section IV, we give a conclusion and show the potential application of the 2-D (M, d, ∞) constraints and codes to high-density optical recording channels.

II. Capacity of 2-D (M, d, ∞) Arrays

The transfer matrix method, which has been used to bound the capacity of binary 2-D RLL constraints [1], is extended to calculate the capacity of 2-D (M, d, ∞) constrains on $m \times n$ arrays in this paper. At first, we briefly review this transfer matrix method.

For an $m \times n$ binary array satisfying 2-D $(1, \infty)$ constraint, it is looked as a long sequence of $m \times 1$ column vectors. Each of these column vectors satisfies the 1-D $(1, \infty)$ constraint, that is to say, it contains no two consecutive ones. The number of different column vectors is well known to be F_{m+1}, the Fibonacci number. These column vectors are used to form all the states of a new directed graph G_m, thus the state transitions in graph G_m represent the concatenations of

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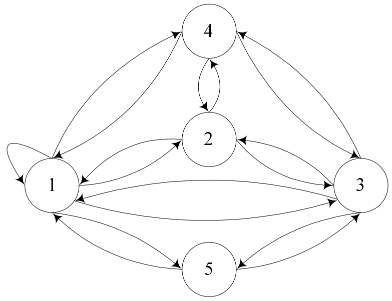

Fig. 1. Directed Graph (FSTD) G_3

column vectors in the $m \times n$ array. The transfer matrix T_m of graph G_m indicates state transitions satisfying the 2-D $(1, \infty)$ constraint: the element t_{ij}=1 if the transition from state i to j is admissible and t_{ij}=0 if it is not admissible.

Obviously we can obtain an $m \times n$ 2-D $(1, \infty)$ constrained array by n walk on the directed graph G_m. The capacity of the 2-D $(1, \infty)$ constraint on $m \times n$ arrays with finite height m $(n \to \infty)$ is given by

$$C_m = \frac{\log \lambda_m}{m} \tag{2}$$

where λ_m is the largest eigenvalue of T_m. It is known [8] that the definition of (2) is an upper bound on the capacity C defined by (1).

Then we discuss the M-ary extension of the transfer matrix method. At first, we consider $d = 1$. For a 2-D $(M, 1, \infty)$ constrained $m \times n$ array, the states of the directed graph G_m are all of the 1-D $(M, 1, \infty)$ constrained $m \times 1$ column vectors, in which the nonzero elements are selected from the alphabet of $\{1, 2, \cdots M - 1\}$. If the state transition from state i to j is admissible, the element t_{ij} of transfer matrix T_m equals to the number of all admissible patterns of state j. Otherwise the element $t_{ij} = 0$.

For example, consider the case of $m = 3$ for a 2-D $(M, 1, \infty)$ constrained $m \times n$ array, there are five states of the directed graph G_3: $S_1 = (0, 0, 0)^T$, $S_2 = (0, 0, X)^T$, $S_3 = (0, X, 0)^T$, $S_4 = (X, 0, 0)^T$, and $S_5 = (X, 0, X)^T$, where X represents the M-ary nonzero symbol with M-1 possible values. The directed graph G_3 is shown in Fig. 1, which is also named finite-state transition-diagram (FSTD). The transfer matrix T_3 can be easily obtained and expressed as follows

$$T_3 = \begin{pmatrix} 1 & a & a & a & a^2 \\ 1 & 0 & a & a & 0 \\ 1 & a & 0 & a & a^2 \\ 1 & a & a & 0 & 0 \\ 1 & 0 & a & 0 & 0 \end{pmatrix} \tag{3}$$

where $a = M - 1$.

From the FSTD G_3 and transfer matrix T_3, we can easily estimate the admissible transitions between each two states. For example, there are $t_{24} = M - 1$ possible edges from state S_2 to S_4, because state S_4 has only one nonzero M-ary symbol with $M - 1$ possible values. In the same way, The total number of edges from state S_2 to S_5 is $t_{25} = (M - 1)^2$, because state S_5 has two nonzero M-ary symbols.

TABLE I

Capacity of $(M, 1, \infty)$ constraints on $m \times n$ arrays

m	M=3	M=4	M=5	M=6	M=7	M=8
2	0.9163	1.1080	1.2557	1.3766	1.4791	1.5684
3	0.8925	1.0800	1.2257	1.3458	1.4484	1.5381
4	0.8804	1.0659	1.2109	1.3311	1.4341	1.5244
5	0.8731	1.0575	1.2022	1.3225	1.4261	1.5171
6	0.8683	1.0520	1.1965	1.3171	1.4211	1.5127
7	0.8648	1.0480	1.1924	1.3134	1.4179	1.5100
8	0.8622	1.0450	1.1895	1.3107	1.4156	1.5082
9	0.8602	1.0427	1.1872	1.3086	1.4140	1.5069
10	0.8586	1.0408	1.1853	1.3071	1.4128	1.5060
11	0.8573	1.0393	1.1838	1.3058	1.4119	1.5054
12	0.8562	1.0381	1.1826	1.3049	1.4112	1.5049

TABLE II

Capacity of $(M, 2, \infty)$ constraints on $m \times n$ arrays

m	M=3	M=4	M=5	M=6	M=7	M=8
2	0.7250	0.8574	0.9570	1.0373	1.1047	1.1629
3	0.6818	0.8067	0.9015	0.9785	1.0434	1.0998
4	0.6636	0.7852	0.8780	0.9537	1.0178	1.0738
5	0.6535	0.7731	0.8644	0.9390	1.0025	1.0580
6	0.6466	0.7647	0.8551	0.9290	0.9921	1.0473
7	0.6416	0.7588	0.8485	0.9221	0.9849	1.0400
8	0.6379	0.7544	0.8436	0.9168	0.9793	1.0345
9	0.6350	0.7509	0.8397	0.9126	0.9751	1.0302
10	0.6327	0.7482	0.8367	0.9094	0.9717	1.0268

According to (2), we can use the eigenvalue of transfer matrix T_m to calculate the capacity of 2-D $(M, 1, \infty)$ constraints on $m \times n$ arrays. Table I shows some numerical results. The height m of 2-D $m \times n$ array varies from 2 to 12, and the parameter M varies from 3 to 8. It can be seen that, when height m is kept constant, the capacity increases quickly with the level parameter M. This demonstrates that larger value of M can easily increase the amount of recorded information. While with the same recording level M, the capacity decreases slowly with the height m, which is because that larger value of m will strengthen the constraints on 2-D arrays.

It should be noted that these numerical results are accurately calculated, and they can be looked as upper bounds on the capacity of general 2-D $(M, 1, \infty)$ constraints on the entire plane. The capacity values listed in Table I give the limitation of information that can be recorded in certain M-ary 2-D recording channels. From constrained coding theory [13], these results can give us directions on the encoder construction of 2-D $(M, 1, \infty)$ constrained codes on $m \times n$ arrays.

Now we consider the case of $d = 2$. For the 2-D $(M, 2, \infty)$ constraints on $m \times n$ arrays, the transfer matrix method

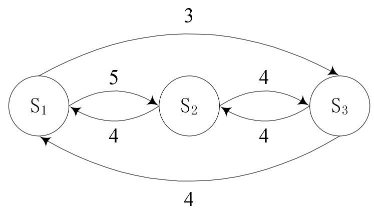

Fig. 2. Final FSTD After State Splitting and Merging

becomes more complex. The $m \times 2$ column vectors satisfying the $(M, 2, \infty)$ constraints form all states $\{S_1, S_2, \cdots, S_N\}$ of the corresponding FSTD. If the second column of S_i is identical to the first column of S_j, we then append the second column of state S_j to the vector S_i and form a new pattern. If the resulted new pattern still satisfies the 2-D $(M, 2, \infty)$ constraints, the transition from S_i to S_j is admissible, and the second column of S_j is generated by the transition. The element t_{ij} of transfer matrix T_m equals to the number of all possible patterns of the second column of state S_j.

Using the eigenvalue of the transfer matrix, we can calculate the capacity of 2-D $(M, 2, \infty)$ constraints on $m \times n$ arrays. Table II shows some numerical results in the case of $d = 2$. The height m of 2-D array is from 2 to 10, and the parameter M varies from 3 to 8. We can easily find that the variation trends of the capacity values are the same as in the case of $d = 1$. For the same parameter of m and M, the capacity of $d = 2$ is obviously smaller than that of $d = 1$. This result can be explained that when the RLL d constraint becomes more strong from $d = 1$ to $d = 2$, fewer information can be recorded and thus the capacity decreases.

When the RLL parameter d grows to be $d > 2$, the complexity of the construction algorithm increases significantly, and the capacity bounds are more difficult to derive via the transfer matrix method. But in a practical optical recording channel, the RLL parameter d is usually limited to be $d = 1$ or $d = 2$ [13]. So the discussion of 2-D (M, d, ∞) constraints on $m \times n$ arrays is sound enough with the consideration of real application. The capacity values derived in this paper can guide us to construct efficient 2-D (M, d, ∞) constrained codes on $m \times n$ arrays.

III. Finite-State Encoder

We now discuss efficient finite-state encoder for the 2-D (M, d, ∞) constraints on $m \times n$ arrays. According to constrained coding theory, the modulation encoder has the task of translating p user bits to q constrained channel symbols [13]. A finite-state encoder can always be constructed if and only if the code rate is chosen to be $R = p/q \leq C$, where C is the capacity of the constraint.

Since the codewords of 2-D (M, d, ∞) constrained codes are not traditional channel bits but 2-D arrays, we should represent the parameter q by $q = mn$. Thus the problem of designing a 2-D modulation encoder is reduced to a one-dimensional coding problem. The finite-state coding theory can be used to design 2-D (M, d, ∞) constrained codes on $m \times n$ arrays.

TABLE III

Encoder table for 2-D $(5, 1, \infty)$ code

User bits	State I Outputs, NS	State II Outputs, NS	State III Outputs, NS
000	$(000)^T$, II	$(000)^T$, III	$(400)^T$, I
001	$(001)^T$, II	$(010)^T$, I	$(001)^T$, II
010	$(002)^T$, II	$(020)^T$, I	$(002)^T$, II
011	$(003)^T$, II	$(030)^T$, I	$(003)^T$, II
100	$(004)^T$, II	$(040)^T$, I	$(004)^T$, II
101	$(100)^T$, III	$(100)^T$, III	$(010)^T$, I
110	$(200)^T$, III	$(200)^T$, III	$(020)^T$, I
111	$(300)^T$, III	$(300)^T$, III	$(030)^T$, I

TABLE IV

Decoder table for 2-D $(5, 1, \infty)$ code

Codewords	User bits	Codewords	User bits
$(000)^T$	000	$(200)^T$	110
$(400)^T$	000	$(300)^T$	111
$(001)^T$	001	$(X00)^T, (010)^T$	$\cdots$, 101
$(002)^T$	010	$(00X)^T, (010)^T$	$\cdots$, 001
$(003)^T$	011	$(X00)^T, (020)^T$	$\cdots$, 110
$(004)^T$	100	$(00X)^T, (020)^T$	$\cdots$, 010
$(040)^T$	100	$(X00)^T, (030)^T$	$\cdots$, 111
$(100)^T$	101	$(00X)^T, (030)^T$	$\cdots$, 011

For an example, we consider a code with parameter $M = 5$, $d = 1$ and $m = 3$, i.e. 2-D $(5, 1, \infty)$ code on $3 \times n$ arrays. The capacity of this 2-D $(5, 1, \infty)$ constraint on $3 \times n$ arrays is 1.2257, as shown in Table I. Because the height of the 2-D array (i.e. the coded array) equals to 3, we can choose a code rate to be $R = 3/(3 \times 1) < 1.2257$. This means that the input of 3 user bits can be transformed to the output of 2-D 3×1 arrays. These 2-D arrays all satisfy the 2-D $(5, 1, \infty)$ constraints even when they are serially concatenated.

In order to construct a finite-state encoder with code rate $R = 3/(3 \times 1)$, we can use the standard state splitting and merging algorithm [14], and the aim is to find a FSTD in which there are at least $2^3 = 8$ edges departed from every state. The original FSTD of the 2-D $(5, 1, \infty)$ constraints is shown in Fig. 1. It can be seen that the edges from state S_5 is only equals to 5, thus we delete this state and obtain a new FSTD with four states. On the next step, we merge state S_1 into state S_2 and S_3. Then the final FSTD is obtained to have only three states, which is shown in Fig. 2. The edges connecting each state in the final FSTD are labelled by the number of admissible transitions used for encoding. In fact, there are at least 8 edges departed from all the sates in the final FSTD.

Now the assignment of codewords is an easy work, and the mapping between input bits and coded array is essentially arbitrary. One possible encoding table is shown in Table III. The rules of state transitions can be easily summarized as follows. The transitions from any other state to S_1 generate vectors as $(0X0)^T$; the transitions from any other state to S_2 generate vectors as $(00X)^T$; and the transitions from any other state to S_1 generate vectors as $(X00)^T$, where X represents the nonzero symbol in the alphabet $\{1,2,3,4\}$. Only one exception occurs when the input user bits include a long sequence of zeros. In this case, a $(400)^T$ vector is produced by the transition from S_3 to S_1, which can be used for timing recovery in the practical recording channel.

The decoding rule is very simple as shown in Table IV. It is easy to see that most coded arrays are block-decodable, which means decoding only relies on the current coded array. If the current coded array is $(0X0)^T$, we need a look-ahead of one coded array for decoding. For example, two consecutive coded arrays as $(300)^T, (020)^T$ are decoded to be $111, 110$, while coded arrays as $(003)^T, (020)^T$ should be decoded to be $011, 010$. The is a sliding-block decoder and the sliding-window is at most to be 2.

Obviously, the proposed finite-state encoder and sliding-block decoder for 2-D $(5, 1, \infty)$ constrained codes are simple and easy for implementation. The coding rate is high of $R = 3/(3 \times 1)$ with efficiency of $\eta = R/C = 81.6\%$. In principle, we can construct some other encoders with higher code rate such as $R = 7/(3 \times 2) < 1.2257$, but the encoder construction process will be much more complex.

IV. Conclusion

We introduce new 2-D (M, d, ∞) constraints in this paper. With the consideration of practical application for advanced optical recording channel, we only discuss the 2-D (M, d, ∞) constraints on $m \times n$ arrays with fixed height m, and RLL parameter d is limited to be $d = 1$ and $d = 2$. The transfer matrix method is used to calculate the capacity of 2-D (M, d, ∞) constraints on $m \times n$ arrays. Some numerical results show the capacity values of 2-D (M, d, ∞) constraints are much higher than that of traditional binary 2-D RLL constraints. Efficient finite-state encoder can be constructed by state splitting and merging algorithm, and we present an example of encoder with code rate of $R = 3/(3 \times 1)$ and efficiency of $\eta = 81.6\%$ for the 2-D $(5, 1, \infty)$ constrained codes on $3 \times n$ arrays.

The proposed 2-D (M, d, ∞) constraints and modulation codes have great potential to be applied in advanced high-density 2-D optical recording channels, such as MWML and TwoDOS storage systems. With the RLL d constraints in horizontal and vertical directions, the 2-D (M, d, ∞) constrained codes may effectively combat the 2-D inter-symbol interference (ISI) in 2-D optical recording channels and improve the system performances. Nevertheless, there are a great deal work need to be developed on the capacity calculation and codes construction of the M-ary 2-D RLL constraints.

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