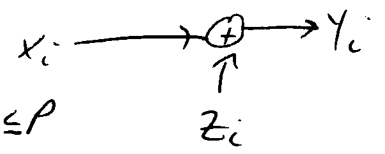
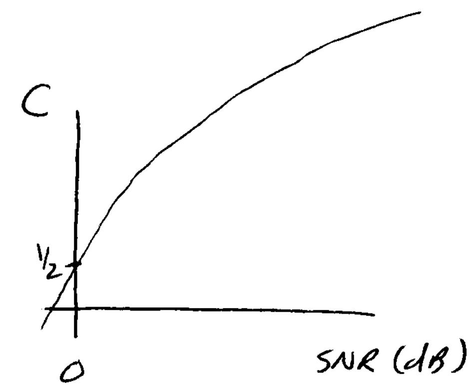
Supplementary material on AWGN channel



$E[X_i^2] \le P$

$Z_i \sim \mathcal{N}(0, N)$ ← mean, variance



$$C = \frac{1}{2} \log_2 \left(1 + \frac{P}{N}\right) \text{ bits/chan use}$$

$\frac{P}{N}$ ↑ SNR

Another viewpoint:

to transmit $R < C$ bits/channel use

$$R < C = \frac{1}{2} \log_2 (1 + SNR)$$

$$2R < \log_2(1 + SNR) \Rightarrow \boxed{SNR > 2^{2R} - 1}$$

6 dB per bit !!

ex: $R = \frac{1}{2}$ bits/channel use

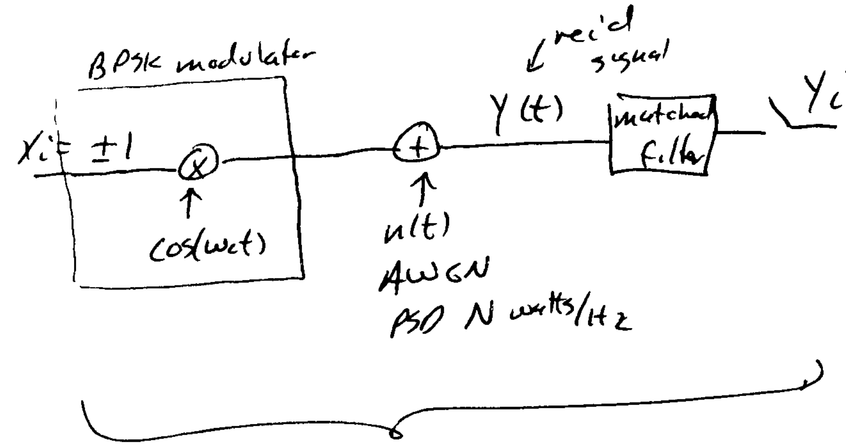
$\Rightarrow SNR > 1 = 0\,dB$

How does one achieve capacity in practice?

Let's consider maximum information rates of some modulation schemes in presence of AWGN.

$\Rightarrow$ see how close they come to ~~cap~~ capacity.

Aside: from a senior level course in digital communications.



$$x_i \in \{\pm 1\} \longrightarrow \oplus \longrightarrow Y_i, \qquad Z_i \sim \mathcal{N}(0, N)$$

(same model whose capacity is $\frac{1}{2}\log\left(1+\frac{P}{N}\right)$ except we have fixed channel input dist'n)

Normally we assume $P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$

End Aside

Since we have fixed the modulation scheme we can compute information transfer of BPSK (often times this is referred to as "capacity of BPSK")

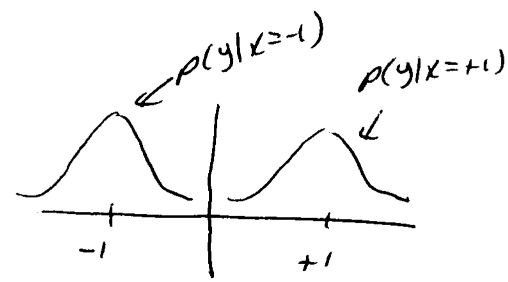
$$I(X;Y) = h(Y) - h(Y|X)$$

for the BPSK channel

we need: $p(y)$ and $p(y|x)$

$$p(y|x=+1) = \frac{1}{\sqrt{2\pi N}} e^{-(y-1)^2/2N}$$

$$p(y|x=-1) = \frac{1}{\sqrt{2\pi N}} e^{-(y+1)^2/2N}$$



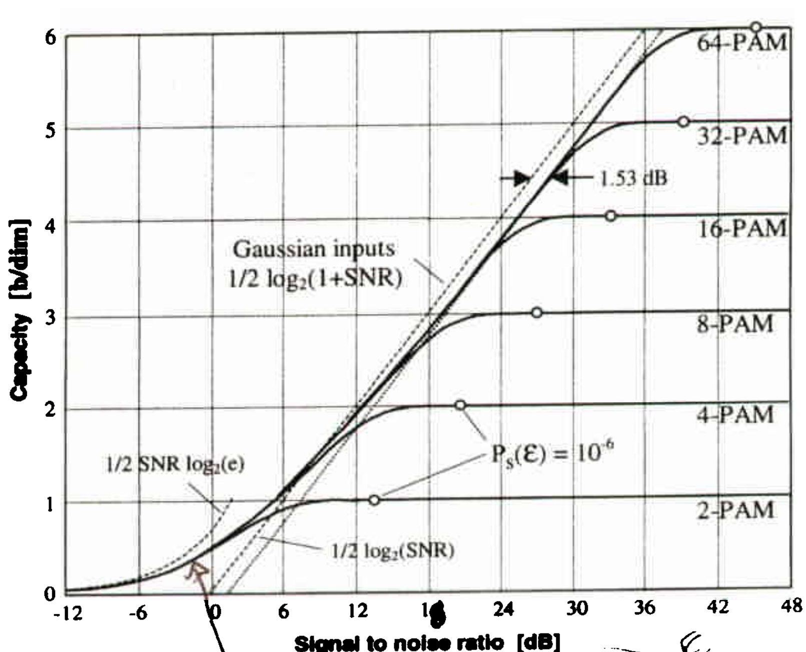
$$p(y) = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi N}}\left(e^{-(y-1)^2/2N} + e^{-(y+1)^2/2N}\right)\right]$$

3

$$h(y) = -\int_y p(y)\log_2 p(y)$$ ← solve numerically it is a function of N (matlab!! :))

$$h(y|x) = -\sum_{x=\pm 1} p(x)\int_y p(y|x)\log_2 p(y|x)$$

$$= \underbrace{-\frac{1}{2}\int_y p(y|x=+1)\log_2 p(y|x=+1)\,dy - \frac{1}{2}\int p(y|x=-1)\log_2 p(y|x=-1)\,dy}_{\text{integrate numerically, its a function of } N \text{ (matlab!! :) :))}}$$



→ E_s.

From *Information Theory: 50 Years of Discovery* edited by Verdú and McLaughlin IEEE Press, 1999.

BSC and AWGN capacity as same at low SNR

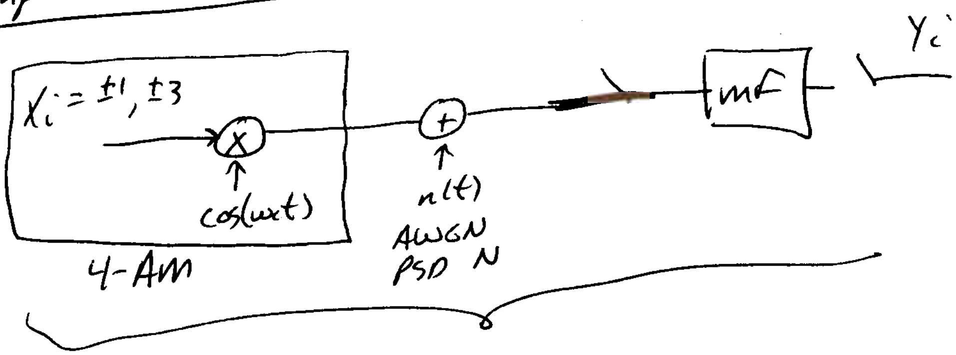
* Note from plot: (1) At low SNR BPSK can be used (in theory) to achieve capacity

(2) At high SNR BPSK is very far from capacity

Examples of low SNR channels: deep space channel

How do we get closer to capacity? Use better modulation scheme: one that transmits more bits/channel use.

Amplitude modulation:



$$X_i \longrightarrow \oplus \longrightarrow Y_i$$

$\pm 1, \pm 3$ $\qquad Z_i \sim N(0, N)$

Assume $\pm 1, \pm 3$ used equally likely: $\quad P = E[X^2] = \frac{1}{4}\left[1^2 + 1^2 + 3^2 + 3^2\right] = 5$

Compute information transfer for 4-AM (Capacity of 4-AM)

$$I(X;Y) = h(y) - h(y|x)$$

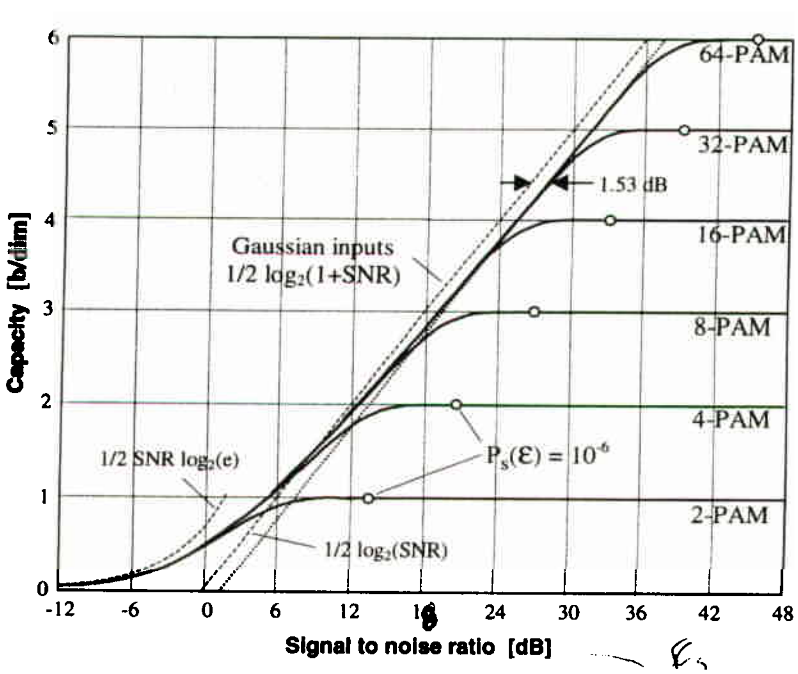
need $p(y)$ $\qquad$ need $p(y|x)$

$\underline{p(y|X)}$:
$$p(y|x=+1) = \frac{1}{\sqrt{2\pi N}} e^{-(y-1)^2/2N}$$
$$p(y|x=-1) = \frac{1}{\sqrt{2\pi N}} e^{-(y+1)^2/2N}$$
$$p(y|x=+3) = \frac{1}{\sqrt{2\pi N}} e^{-(y-3)^2/2N}$$
$$p(y|x=-3) = \frac{1}{\sqrt{2\pi N}} e^{-(y+3)^2/2N}$$

$\underline{p(y)}$:
$$p(y) = \sum_x p(y|x)\,p(x) = \frac{1}{4}\Big[p(y|x=1) + p(y|x=3) + p(y|x=-1) + p(y|x=-3)\Big]$$

Compute: $I(X;Y) = h(y) - h(y|x)$
$$= \underbrace{-\int_y p(y)\log_2 p(y)\,dy + \frac{1}{4}\sum_{x=\pm1,\pm3}\int_y p(y|x)\log_2 p(y|x)\,dy}_{\text{numerically integrate using YKW (you know what)}}$$

↑ function of N only



(*) Previous plots showed information transfer of AM schemes.

⇒ what is actual performance of AM schemes?

⇒ For now let's assume that good performance means $p(\omega \neq \hat{\omega}) = 10^{-6}$

⇒ It is possible to show that the error probability as a function of SNR for BPSK is (take any digital communications course)

$$p(\omega \neq \hat{\omega}) = Q\left(\sqrt{\frac{P}{N}}\right)$$

$$\text{where } Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

(*) Note also gap between AM plots and capacity. This is the loss of using uniform input distribution instead of gaussian (shaping loss).

(*) It is possible to show that for M-ary AM: $p(\omega \neq \hat{\omega}) = 2Q\left(\sqrt{\frac{d^2}{4N}}\right)$ where d^2 is squared distance between points in signal set of AM points.

# of nearest neighbors in signal set

So BPSK is a long way from capacity
4-AM is " " " " "

How